Uniform Grid Numerical Scheme (UGNS) For Solving Scientific and Engineering Problems Modelled in Ordinary Differential Equations

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Abstract
This study presents the development and numerical analysis of a Uniform Grid Numerical Scheme (UGNS) for solving first-order ordinary differential equations (ODEs), with particular emphasis on stiff initial value problems arising in applied science and engineering. The scheme is formulated using a power series polynomial as the basis function, employing interpolation and collocation procedures based on second derivative approximations to derive a continuous linear multistep method. The resulting discrete scheme is obtained through Gaussian elimination and evaluated at off-grid points to yield the UGNS. The basic properties of the scheme are rigorously analyzed: it is of uniform order six with a specified error constant, consistent as verified through the first and second characteristic polynomials and zero-stable according to the root condition. These properties collectively satisfy Dahlquist's theorem, establishing the convergence of the scheme. The region of absolute stability is determined. The performance of the UGNS is validated on three benchmark problems: pseudo-first-order hydrolysis of ethyl acetate from industrial chemistry, an RC electrical circuit from applied physics and a stiff system of differential equations from applied mathematics. Numerical results demonstrate excellent agreement with analytical solutions, with the scheme achieving machine-precision accuracy for the ester hydrolysis problem. The RC circuit results exhibit physically consistent transient behavior, though lacking quantitative error analysis due to the absence of an analytical solution. Graphical comparisons confirm that the numerical and analytical curves are visually indistinguishable. The findings establish the UGNS as a highly accurate, stable and versatile numerical tool for solving ODEs, with particular strength in handling stiff systems where conventional methods often experience stability difficulties. The scheme's ability to maintain machine-precision accuracy across diverse problem types and solution behaviors underscores its reliability and computational efficiency for real-world applications in science and engineering.

Keywords: *Uniform Grid Numerical Scheme, second derivative methods, stiff ordinary differential equations, convergence analysis, absolute stability, collocation method*

1 Introduction

Chemical kinetics, fluid dynamics, heat transport, population dynamics, and electrical circuits are just a few of the real-world issues in applied science, engineering, and technology that are frequently modeled using ordinary differential equations (ODEs) (Fotta & Alabi, 2015; Donald et al., 2026). Numerical techniques that yield precise, reliable, and computationally efficient solutions are becoming more and more necessary because many of these models cannot be solved

analytically. However, in order to preserve stability, many traditional numerical approaches demand very tiny step sizes, which raises computing costs, especially for stiff issues (Fogler, 2016; Donald et al., 2022; Muhammad et al., 2026).

The goal of the Uniform Grid Numerical Scheme (UGNS) is to decrease error propagation while increasing numerical solution accuracy, stability, and computational economy. It is ideal for resolving real-world issues in applied research and engineering because of its theoretical characteristics, which guaranty accurate approximations over uniformly spaced grid points (Abdelrahim et al., 2016; Rufai et al., 2016). Benchmark initial value problems with known analytical solutions, such as pseudo-first-order hydrolysis of ethyl acetate and radioactive decay, were used to validate the scheme's performance. The results showed superior computational accuracy and excellent agreement with the exact solutions (Atkins & de Paula, 2014; Chapra & Canale, 2015; Fogler, 2016). These findings demonstrate the UGNS dependability and effectiveness as a numerical instrument for resolving a wide range of modeled applied science issues controlled by ordinary differential equations.

This study considers the numerical solution of real life initial value problems modeled as first order ordinary differential equation of the form

$$\frac{dy}{dx} = f(x, y(x)), \quad y(x_1) = \rho \tag{1.1}$$

In the numerical solution of ordinary differential equations (ODEs), UGNS have become a significant development, especially for stiff systems found in science and engineering (Donald, 2022). These techniques enhance numerical accuracy, convergence, and error control without incurring undue processing costs by integrating second derivative information into traditional block hybrid formulations (Ibrahim et al., 2019). They are more efficient in handling stiff behavior than conventional first-derivative-based techniques because they can compute several solution points at once in a single integration step (Abhulimen & Igbokwe, 2017; Olatunji, 2019; Olabode & Olatunji, 2024). As a result, EESNS are widely used in fields where precise and stable numerical solutions are crucial, such as chemical kinetics, electrical circuit simulation, control systems, systems biology, pharmacokinetics, fluid flow, heat transfer, epidemiology, population dynamics, and aerospace engineering (Skwame et al., 2017; Oni et al., 2021; Panopoulos et al., 2023; Yakubu et al., 2023). The development of second derivative numerical techniques for solving stiff ordinary differential equations has advanced significantly, with ongoing gains in processing efficiency, accuracy, stability, and convergence. Numerical schemes with greater regions of absolute stability, lower error constants, and lower computational costs have been produced by the introduction of second derivative multistep block methods, multiderivative collocation techniques, blended block methods, implicit hybrid formulations, and backward differentiation approaches (Tumba et al., 2018; Skwame et al., 2018; Donald et al., 2022). These advancements have made second derivative block hybrid methods dependable and effective numerical approaches for solving stiff initial value problems that arise in engineering and scientific applications.

2 Numerical Formulation of Uniform Grid Numerical Scheme (UGNS)

In this part, the Uniform Grid Numerical Scheme (UGNS) was numerically formulated utilizing the power series polynomial as a fundamental function. Using a second derivative numerical approach, an interpolation and collocation procedure was used to derive a UGNS.

Let the power series polynomial as a basic function of the form

$$y(x)=\sum_{j=0}^{r+s-1}\Omega\, x^{j} \tag{2.1}$$

be the approximate solution to equation (1.1) for $x\in[x_n,x_{n+2}]$ where $r+s$ is the sum of the number of interpolation and collocation points, $n=0,\ 1,\ 2,\ \ldots,\ N-1,\ a's$ are the real coefficients to be determined, $h=x_n-x_{n-1}$ is a constant step size of the partition of interval $[u,\ v]$. We assume that the off-grid point under consideration will be $0,\ p,\ \cdots,1,\ q,\ \cdots,\ 2$.

Differentiating equation (2.2) once and twice yield to get

$$\frac{dy}{dx}=\sum_{i=0}^{r+s-1}j\Omega\, x^{j-1} \tag{2.2}$$

$$\frac{d^2y}{dx^2}=\sum_{i=0}^{r+s-1}j(j-1)\Omega\, x^{j-2} \tag{2.3}$$

Now, interpolate equation (2.1) at $x_{n+r},\ r=0$ and collocate equations (2.2) at $x_{n+s},\ s=1,\ 2$ and (2.3) at $x_{n+s},\ s=0\left(\frac{1}{5}\right)2$, which leads to the following system of equation in a matrix form as

$$\begin{bmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 & x_n^9 & x_n^{10} & x_n^{11} & x_n^{12} & x_n^{13}\\
0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 & 6x_{n+1}^5 & 7x_{n+1}^6 & 8x_{n+1}^7 & 9x_{n+1}^8 & 10x_{n+1}^9 & 11x_{n+1}^{10} & 12x_{n+1}^{11} & 13x_{n+1}^{12}\\
0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & 5x_{n+2}^4 & 6x_{n+2}^5 & 7x_{n+2}^6 & 8x_{n+2}^7 & 9x_{n+2}^8 & 10x_{n+2}^9 & 11x_{n+2}^{10} & 12x_{n+2}^{11} & 13x_{n+2}^{12}\\
0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^6 & 56x_n^6 & 72x_n^7 & 90x_n^8 & 110x_n^9 & 132x_n^{10} & 156x_n^{11}\\
0 & 0 & 2 & 6x_{n+\frac{1}{5}} & 12x_{n+\frac{1}{5}}^2 & 20x_{n+\frac{1}{5}}^3 & 30x_{n+\frac{1}{5}}^4 & 42x_{n+\frac{1}{5}}^6 & 56x_{n+\frac{1}{5}}^6 & 72x_{n+\frac{1}{5}}^7 & 90x_{n+\frac{1}{5}}^8 & 110x_{n+\frac{1}{5}}^9 & 132x_{n+\frac{1}{5}}^{10} & 156x_{n+\frac{1}{5}}^{11}\\
0 & 0 & 2 & 6x_{n+\frac{2}{5}} & 12x_{n+\frac{2}{5}}^2 & 20x_{n+\frac{2}{5}}^3 & 30x_{n+\frac{2}{5}}^4 & 42x_{n+\frac{2}{5}}^6 & 56x_{n+\frac{2}{5}}^6 & 72x_{n+\frac{2}{5}}^7 & 90x_{n+\frac{2}{5}}^8 & 110x_{n+\frac{2}{5}}^9 & 132x_{n+\frac{2}{5}}^{10} & 156x_{n+\frac{2}{5}}^{11}\\
0 & 0 & 2 & 6x_{n+\frac{3}{5}} & 12x_{n+\frac{3}{5}}^2 & 20x_{n+\frac{3}{5}}^3 & 30x_{n+\frac{3}{5}}^4 & 42x_{n+\frac{3}{5}}^6 & 56x_{n+\frac{3}{5}}^6 & 72x_{n+\frac{3}{5}}^7 & 90x_{n+\frac{3}{5}}^8 & 110x_{n+\frac{3}{5}}^9 & 132x_{n+\frac{3}{5}}^{10} & 156x_{n+\frac{3}{5}}^{11}\\
0 & 0 & 2 & 6x_{n+\frac{4}{5}} & 12x_{n+\frac{4}{5}}^2 & 20x_{n+\frac{4}{5}}^3 & 30x_{n+\frac{4}{5}}^4 & 42x_{n+\frac{4}{5}}^6 & 56x_{n+\frac{4}{5}}^6 & 72x_{n+\frac{4}{5}}^7 & 90x_{n+\frac{4}{5}}^8 & 110x_{n+\frac{4}{5}}^9 & 132x_{n+\frac{4}{5}}^{10} & 156x_{n+\frac{4}{5}}^{11}\\
0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 & 42x_{n+1}^6 & 56x_{n+1}^6 & 72x_{n+1}^7 & 90x_{n+1}^8 & 110x_{n+1}^9 & 132x_{n+1}^{10} & 156x_{n+1}^{11}\\
0 & 0 & 2 & 6x_{n+\frac{6}{5}} & 12x_{n+\frac{6}{5}}^2 & 20x_{n+\frac{6}{5}}^3 & 30x_{n+\frac{6}{5}}^4 & 42x_{n+\frac{6}{5}}^6 & 56x_{n+\frac{6}{5}}^6 & 72x_{n+\frac{6}{5}}^7 & 90x_{n+\frac{6}{5}}^8 & 110x_{n+\frac{6}{5}}^9 & 132x_{n+\frac{6}{5}}^{10} & 156x_{n+\frac{6}{5}}^{11}\\
0 & 0 & 2 & 6x_{n+\frac{7}{5}} & 12x_{n+\frac{7}{5}}^2 & 20x_{n+\frac{7}{5}}^3 & 30x_{n+\frac{7}{5}}^4 & 42x_{n+\frac{7}{5}}^6 & 56x_{n+\frac{7}{5}}^6 & 72x_{n+\frac{7}{5}}^7 & 90x_{n+\frac{7}{5}}^8 & 110x_{n+\frac{7}{5}}^9 & 132x_{n+\frac{7}{5}}^{10} & 156x_{n+\frac{7}{5}}^{11}\\
0 & 0 & 2 & 6x_{n+\frac{8}{5}} & 12x_{n+\frac{8}{5}}^2 & 20x_{n+\frac{8}{5}}^3 & 30x_{n+\frac{8}{5}}^4 & 42x_{n+\frac{8}{5}}^6 & 56x_{n+\frac{8}{5}}^6 & 72x_{n+\frac{8}{5}}^7 & 90x_{n+\frac{8}{5}}^8 & 110x_{n+\frac{8}{5}}^9 & 132x_{n+\frac{8}{5}}^{10} & 156x_{n+\frac{8}{5}}^{11}\\
0 & 0 & 2 & 6x_{n+\frac{9}{5}} & 12x_{n+\frac{9}{5}}^2 & 20x_{n+\frac{9}{5}}^3 & 30x_{n+\frac{9}{5}}^4 & 42x_{n+\frac{9}{5}}^6 & 56x_{n+\frac{9}{5}}^6 & 72x_{n+\frac{9}{5}}^7 & 90x_{n+\frac{9}{5}}^8 & 110x_{n+\frac{9}{5}}^9 & 132x_{n+\frac{9}{5}}^{10} & 156x_{n+\frac{9}{5}}^{11}\\
0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 & 42x_{n+2}^6 & 56x_{n+2}^6 & 72x_{n+2}^7 & 90x_{n+2}^8 & 110x_{n+2}^9 & 132x_{n+2}^{10} & 156x_{n+2}^{11}
\end{bmatrix}
\begin{bmatrix}a_0\\a_1\\a_2\\a_3\\a_4\\a_5\\a_6\\a_7\\a_8\\a_9\\a_{10}\\a_{11}\\a_{12}\\a_{13}\end{bmatrix}=
\begin{bmatrix}y_n\\f_{n+1}\\f_{n+2}\\g_n\\g_{n+\frac{1}{5}}\\g_{n+\frac{2}{5}}\\g_{n+\frac{3}{5}}\\g_{n+\frac{4}{5}}\\g_{n+1}\\g_{n+\frac{6}{5}}\\g_{n+\frac{7}{5}}\\g_{n+\frac{8}{5}}\\g_{n+\frac{9}{5}}\\g_{n+2}\end{bmatrix} \tag{2.4}$$

Solving the values of $a_{j's}$ in equation (2.4) using Gaussian elimination method, the values of the $a_{j's}$ obtain are substituted into (2.1) to give a continuous form of UGNS in the form of the continuous linear multistep method as

$$y(x)=\alpha_0(x)y_n+h[\beta_1(x)f_{n+1}+\beta_2(x)f_{n+2}]+$$
$$h^2\begin{bmatrix}\gamma_0(x)g_n+\gamma_{\frac{1}{5}}(x)g_{n+\frac{1}{5}}+\gamma_{\frac{2}{5}}(x)g_{n+\frac{2}{5}}+\gamma_{\frac{3}{5}}(x)g_{n+\frac{3}{5}}+\gamma_{\frac{4}{5}}(x)g_{n+\frac{4}{5}}+\gamma_1(x)g_{n+1}\\+\gamma_{\frac{6}{5}}(x)g_{n+\frac{6}{5}}+\gamma_{\frac{7}{5}}(x)g_{n+\frac{7}{5}}+\gamma_{\frac{8}{5}}(x)g_{n+\frac{8}{5}}+\gamma_{\frac{9}{5}}(x)g_{n+\frac{9}{5}}+\gamma_2(x)g_{n+2}\end{bmatrix} \tag{2.5}$$

Evaluate equation (2.4) at $\frac{1}{5}\left(\frac{1}{5}\right)2$ to obtain the UGNS as

$$\begin{bmatrix} y_{n+\frac{1}{5}} \\ y_{n+\frac{2}{5}} \\ y_{n+\frac{3}{5}} \\ y_{n+\frac{4}{5}} \\ y_{n+1} \\ y_{n+\frac{6}{5}} \\ y_{n+\frac{7}{5}} \\ y_{n+\frac{8}{5}} \\ y_{n+\frac{9}{5}} \\ y_{n+2} \end{bmatrix} = y_n + h \begin{bmatrix} \frac{1569102436}{1149017187\,5} & \frac{728931939}{1149017187\,5} \\ \frac{3936795392}{1149017187\,5} & \frac{659273358}{1149017187\,5} \\ \frac{6201750258}{1149017187\,5} & \frac{692352867}{1149017187\,5} \\ \frac{8489304064}{1149017187\,5} & \frac{702833436}{1149017187\,5} \\ \frac{3446656}{3676855} & \frac{230199}{3676855} \\ \frac{1305229593\,6}{1149017187\,5} & \frac{735910314}{1149017187\,5} \\ \frac{2191407106}{1641453125} & \frac{106627269}{1641453125} \\ \frac{1760480460\,8}{1149017187\,5} & \frac{779470392}{1149017187\,5} \\ \frac{1997249756\,4}{1149017187\,5} & \frac{709811811}{1149017187\,5} \\ \frac{6893312}{3676855} & \frac{460398}{3676855} \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+2} \end{bmatrix} +$$

$$h^2 \begin{bmatrix} -\frac{7778384535\,0943}{3386960438\,4000000} & -\frac{3797095631\,79827}{8806097139\,840000} & -\frac{5609227930\,7}{2823824640\,000} & -\frac{6020658809\,6951}{7338414283\,20000} & \frac{2327285853\,9677}{9784552377\,60000} & -\frac{1255129123\,676873}{1223069047\,2000000} \\ -\frac{1477594616\,83}{6615157106\,2500} & -\frac{1588364726\,749}{3439881695\,2500} & -\frac{1925801599\,79}{3276077805\,000} & -\frac{1329253411\,7}{1146627231\,75} & -\frac{7801126423\,9}{2866568079\,375} & -\frac{1622055485\,3453}{1433284039\,68750} \\ -\frac{2531991823\,4293}{1128986812\,8000000} & -\frac{1350212079\,95281}{2935365713\,280000} & -\frac{1747539703\,7633}{2795586393\,60000} & -\frac{3726790503\,2381}{2446138094\,40000} & -\frac{2689863095\,3}{3953354496\,00} & -\frac{1667094177\,220169}{1223069047\,2000000} \\ -\frac{1483536385\,61}{6615157106\,2500} & -\frac{3956214150\,89}{8599704238\,125} & -\frac{3784870907}{6066810750\,0} & -\frac{4462757189\,96}{2866568079\,375} & -\frac{4074133747\,9}{3822090772\,50} & -\frac{3752365546\,246}{2388806732\,8125} \\ -\frac{2432704192\,3}{1083827340\,2880} & -\frac{6478003491\,55}{1408975542\,3744} & -\frac{7634111545}{1219892244\,48} & -\frac{1816443234\,95}{1174146285\,312} & -\frac{5311900142\,65}{4696585141\,248} & -\frac{2041706880\,079}{1174146285\,3120} \\ -\frac{4952351611\,3}{2205052368\,7500} & -\frac{5269675963\,19}{1146627231\,7500} & -\frac{9781887919}{1560037050\,00} & -\frac{1473122550\,07}{9555226931\,25} & -\frac{1221005177\,3}{1061691881\,25} & -\frac{6911725230\,53}{4343284968\,750} \\ -\frac{1553404896\,079}{6912164160\,00000} & -\frac{8255791044\,131}{1797162681\,60000} & -\frac{2788476926\,71}{4437438720\,000} & -\frac{4599096622\,03}{2995271136\,000} & -\frac{2344346802\,643}{1996847424\,0000} & -\frac{3486021200\,9369}{2496059280\,00000} \\ -\frac{7438932772\,1}{3307578553\,1250} & -\frac{3948523688\,02}{8599704238\,125} & -\frac{5159216236\,1}{8190194512\,50} & -\frac{4382368296\,08}{2866568079\,375} & -\frac{1373983927\,3}{1146627231\,75} & -\frac{8645364572\,836}{7166420198\,4375} \\ -\frac{8467381499\,623}{3763289376\,000000} & -\frac{4491180737\,7499}{9784552377\,60000} & -\frac{5879456613\,227}{9318621312\,0000} & -\frac{1242490509\,4079}{8153793648\,0000} & -\frac{3534324583\,7}{2899126630\,40} & -\frac{4135341044\,17859}{4076896824\,000000} \\ -\frac{47633863}{2116850274\,0} & -\frac{2526064685}{5503810712\,4} & -\frac{12255745}{194137944} & -\frac{697944365}{4586508927} & -\frac{187992565}{1528836309} & -\frac{1366384531}{1528836309\,0} \end{bmatrix} \begin{bmatrix} g_n \\ g_{n+\frac{1}{5}} \\ g_{n+\frac{2}{5}} \\ g_{n+\frac{3}{5}} \\ g_{n+\frac{4}{5}} \\ g_{n+1} \\ g_{n+\frac{6}{5}} \\ g_{n+\frac{7}{5}} \\ g_{n+\frac{8}{5}} \\ g_{n+\frac{9}{5}} \\ g_{n+2} \end{bmatrix}$$

$$\begin{bmatrix} \frac{5505439076\,3293}{9784552377\,60000} & -\frac{4251274028\,5879}{7338414283\,20000} & \frac{2019172987\,5641}{1956910475\,520000} & -\frac{1983987074\,93683}{8806097139\,840000} & -\frac{2998006432\,87351}{8806097139\,8400000} \\ \frac{6085392216\,23}{1146627231\,7500} & -\frac{5055888458\,3}{9555226931\,25} & \frac{2164042633\,3}{2293254463\,500} & -\frac{6376056696\,7}{3127165177\,500} & -\frac{1058853238\,847}{3439881695\,25000} \\ \frac{6305989327\,733}{1087172486\,40000} & -\frac{1374692621\,5613}{2446138094\,40000} & \frac{1967870880\,9913}{1956910475\,520000} & -\frac{1258480083\,6893}{5870731426\,56000} & -\frac{9484072388\,0381}{2935365713\,2800000} \\ \frac{1169449141\,97}{1911045386\,250} & -\frac{1652647268\,68}{2866568079\,375} & \frac{3952316176\,3}{3822090772\,500} & -\frac{681113003}{3127165177\,5} & -\frac{5638795296\,17}{1719940847\,62500} \\ \frac{3022307697\,35}{4696585141\,248} & -\frac{2328572916\,5}{3913820951\,04} & \frac{1004761790\,35}{9393170282\,496} & -\frac{3143457571\,55}{1408975542\,3744} & -\frac{9452445829}{2817951084\,7488} \\ \frac{3062378595\,7}{4246767525\,00} & -\frac{5903737738\,3}{9555226931\,25} & \frac{4251594364\,3}{3822090772\,500} & -\frac{2619892986\,71}{1146627231\,7500} & -\frac{3932585827\,57}{1146627231\,75000} \\ \frac{2225666349\,229}{1996847424\,0000} & -\frac{8071778050\,1}{1361486880\,000} & \frac{8903877330\,1}{7987389696\,000} & -\frac{4163410869\,283}{1797162681\,60000} & -\frac{6251845050\,343}{1797162681\,600000} \\ \frac{4437173039\,99}{2866568079\,375} & -\frac{2434596792\,8}{9555226931\,25} & \frac{7995623411}{5211941962\,50} & -\frac{4190756626\,6}{1719940847\,625} & -\frac{3116685014\,57}{8599704238\,1250} \\ \frac{7266432336\,167}{3623908288\,0000} & -\frac{1135102926\,913}{8153793648\,0000} & \frac{3159410657\,177}{5930031744\,0000} & -\frac{3748864046\,303}{1956910475\,52000} & -\frac{3289305720\,4031}{9784552377\,600000} \\ \frac{1418396365}{6115345236} & -\frac{15944785}{416955357} & \frac{510171595}{6115345236} & -\frac{79049315}{5503810712\,4} & -\frac{492025379}{1100762142\,48} \end{bmatrix} \quad (2.6)$$

3 The Numerical Analysis of Uniform Grid Numerical Scheme (UGNS)

The analysis of the basic properties of Uniform Grid Numerical Scheme (UGNS) are analyzed in this section. These properties include; order and error constant, consistency and zero-stability, which reveal the nature of convergence and the regions of absolute stability.

3.1 Order and Error Constant of UGNS

Definition 3.1:

Consider the linear operator associated with the UGNS be defined as

$$\ell[y(x),h]=\sum_{j=0}^{k}\left[(a_j y(x+jh)-h\beta_j y'(x+jh)-h\gamma_j y''(x+jh)\right] \tag{3.1}$$

where $y(x)$ is an arbitrary test function that is continuously differentiable in the interval $[a,b]$. We expand equation (3.1) as a Taylor series about x_n and collecting like terms in h and y to obtain the expression;

$$\ell\{y(x);h\}=C_0y(x)+C_1y'(x)+\ldots+C_ph^py^p(x)+C_{p+1}h^{p+1}y^{p+1}(x)+C_{p+2}h^{p+2}y^{p+2}(x)+\ldots \tag{3.2}$$

The UGNS and the associated linear difference operators are said to have order p if $C_0=C_1=\ldots=C_p=C_{p+1}=C_{p+2}=0, C_{p+3}\neq 0$ (Skwame *et al.*, 2018). The order is also defined as the largest positive real number p that quantifies the rate of convergence of a numerical approximation of a differential equation to that of the exact solution. The term C_{p+3} is called the error constant and implies that the local truncation error for UGNS is given by,

$$T_{n+k}=C_{p+3}h^{p+3}y^{p+3}(t)+O(h^{p+4}) \tag{3.3}$$

Therefore by definition 3.1, the UGNS is of uniform order six and the error constant

$$C_9=\left[-\frac{3250433}{1503157776\,2406374400},\ -\frac{8501}{4697368050\,7519920},\ -\frac{11899}{6185834470\,1260800},\ -\frac{547}{2935855031\,719995},\ -\frac{8712}{8550317199\,95},\ -\frac{114985}{6012631104\,96254976},\ -\frac{179}{9665366359\,57200},\ -\frac{84427}{4294736503\,54467840},\ -\frac{2368}{1467927515\,8599975},\ -\frac{653}{6538673171\,09295}\right]^T.$$

3.2 Consistency of UGNS

Definition 3.2: First and Second Characteristic Polynomials

Given the UGNS, the first and second characteristic polynomials are defined as,

$$\rho(z)=\sum_{j=0}^{k}\alpha_j z^j \tag{3.4}$$

and

$$\sigma(z)=\sum_{j=0}^{k}\beta_j z^j \tag{3.5}$$

where z is the principal root, $\alpha_k\neq 0$ and $\alpha_0^2+\beta_0^2\neq 0$.

It is important to note that consistency controls the magnitude of the local truncation error committed at each stage of the computation. The UGNS is said to be consistent if it satisfies the following conditions;

i. the order $p\geq 1$,

ii. $\sum_{j=0}^{k}\alpha_j=0$, and

iii. $\rho'(1)=\sigma(1)$

Hence, by definition 3.2, UGNS is consistent, since it is of uniform order seven (See Tumba *et al.*, 2019).

3.3 Zero Stability of UGNS

Definition 3.3: The UGNS is said to be zero-stable if as $h \to 0$, the root $z_j, j = 1(1)k$ of the first characteristic polynomial $\rho(z) = 0$ that is $\rho(z) = \det\left[\sum_{j=0}^{k} A^{(j)} z^{k-j}\right] = 0$ Satisfies $|z_j| \le 1$ and for those roots with $|z_j| = 1$, multiplicity must not exceed two. The UGNS is expressed in the form

$$\rho(z) = \left| z \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{pmatrix} - \begin{pmatrix} 1&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&1 \end{pmatrix} \right| = z^9(z-1) \tag{3.6}$$

$$\rho(z) = z^9(z-1) = 0, \quad z = 0, 0, 0, 0, 0, 0, 0, 0, 0, 1$$

Therefore, the UGNS is zero-stable (Skwame *et al.*, 2017).

3.4 Convergence of UGNS

The convergence of the UGNS is considered in the light of the basic properties discussed earlier in conjunction with the fundamental theorem of Dahlquist, for UGNS. In what follows, we state Dahlquist's theorem without proof.

Theorem 1:

The necessary and sufficient condition for a UGNS to be convergent is that, if must be consistent and zero stable, hence our UGNS is convergent (Skwame *et al.*, 2017).

3.5 Region of Absolute Stability of UGNS

Definition 5: The region of absolute stability is a region in the complex z plane, where $z = \lambda h$. It is defined as those values of z such that the numerical solutions of $y'' = -\lambda^2 y$ satisfy $y_j \to 0 \; as \; j \to \infty$ for any initial condition (Skwame *et al.*, 2017). To determine the regions of absolute stability of an algorithm, a method that requires neither the computation of roots of a polynomial nor solving of simultaneous inequalities was adopted. This method is called the Boundary Locus Method (BLM).

Applying the boundary locus method on UGNS, we obtain the stability polynomial for UGNS as

$$\left.\begin{aligned} \bar{h}(w) = {} & h^{10}\left(\frac{1275380352\,63931}{7449432883\,200000}w^9 - \frac{58777}{4204890}w^{10}\right) - h^9\left(-\frac{1021842064\,371587}{6337191168\,0000000000\,0000}w^9 + \frac{183517}{4752}w^{10}\right) + \\ & h^8\left(-\frac{1240787094\,942223}{4055802347\,5200000000\,000}w^9 + \frac{66528}{1181437}w^{10}\right) + h^7\left(\frac{1338797113\,1347}{1469604310\,4784384000\,00}w^9 - \frac{1303}{3214155168}w^{10}\right) + \\ & h^6\left(-\frac{2675675852\,951287}{9797362069\,8562560000\,0}w^9 + \frac{502211745}{4523}w^{10}\right) + h^5\left(\frac{2174780359\,4142223}{1216740704\,2560000000\,0000}w^9 - \frac{985}{5629856}w^{10}\right) + \\ & h^4\left(-\frac{1554228255\,89737}{5431878144\,0000000000\,0000}w^9 - \frac{986}{74640}w^{10}\right) + h^3\left(\frac{1618099577\,79906749}{3910952263\,6800000000\,00}w^9 - \frac{867}{90875}w^{10}\right) + \\ & h^2\left(\frac{1441155772\,9}{1896846336\,0000000000\,00}w^9 + \frac{332}{8524}w^{10}\right) + h\left(\frac{2400000}{10886400}w^9 - \frac{1}{8}w^{10}\right) - w^8 + w^9 \end{aligned}\right\} \tag{3.7}$$

Using the Matlab software, the Absolute Stability Region of UGNS is plotted and is shown below in figure 1.

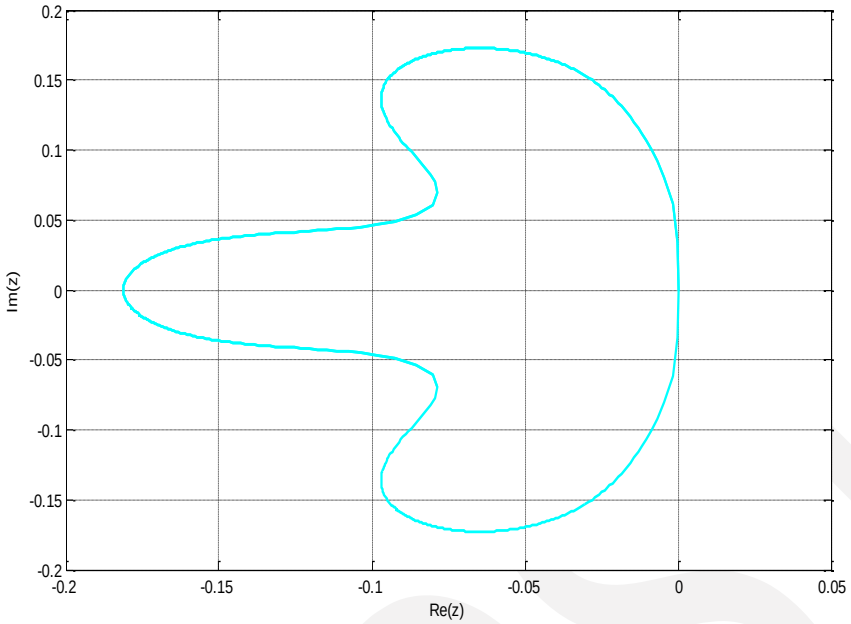

Figure 3.1: Region of Absolute Stability of UGNS.

4 Numerical Application of Uniform Grid Numerical Scheme (UGNS)

The numerical application of Uniform Grid Numerical Scheme (UGNS) was tested on several real life problems (specifically, problems from industrial chemistry, applied Physics and Applied Mathematics) and the results was tabulated and graphically shown to illustrate the performance of UGNS.

Problem 4.1: (Ester Hydrolysis)

Consider the chemical reaction, which is Ethyl acetate hydrolysis (Pseudo-first order)

$$CH_3COOC_2H_5 + H_2O \rightarrow product \tag{4.1}$$

The governing differential equation gives

$$\frac{dc}{dx} = -kCE \tag{4.2}$$

Where CE is the concentration of ester (mol/l),

$k = 0.3\min^{-1}$ (rate of constant) and $c(0) = 1.0\ mol/l$.

The analytical solution is

$$c(x) = \exp(0.3x) \tag{4.3}$$

Source: [Atkins & de Paula, (2014), Fogler, (2016)].

Problem 4.2: (Electrical Circuit)

Consider an RC-Electrical Circuit (Electromagnetism), whose resistor R and capacitor C are connected in series to a battery of *emf* E . The governing differential equation is given by

$$R\frac{dq}{dx}+\frac{q}{c}=E \tag{4.4}$$

If $R=2\Omega, C=0.5F, E=10V$, determine the changes on the capacitor as a function of time x with initial condition $q(x)=0$.

Source: [Boylestad & Nashelsky, 2013, Nilsson & Riedel, 2015]

Problem 4.3: (Stiff Differential Equation)

Consider the stiffly systems of equation of the form

$$\left.\begin{aligned}\frac{dy_1}{dx}&=198y_1+199y_2;\ y_1(0)=1\\ \frac{dy_2}{dx}&=-398y_1-399y_2;\ y_2(0)=-1\end{aligned}\right\} \tag{4.5}$$

With analytic solution of the form

$$\left.\begin{aligned}y_1(x)&=\exp(-x)\\ y_2(x)&=-\exp(-x)\end{aligned}\right\} \tag{4.6}$$

Source, [Skwame, *et al.*, (2017), Skwame, 2018].

The following notations was used in the tables and figures.

Notations	Meaning
x	Point of Evaluation/time
AB	Analytic Solution
NS	Numerical Solution
CE	Concentration of Ester
CC	Change on the Capacitor in UGNS
AEUGNS	Absolute Error in UGNS
AES17	Absolute Error in Skwame *et al.*, (2017)
AES18	Absolute Error in Skwame *et al.*, (2017)

Table 4.1: Numerical Result of Ester Hydrolysis (Problem 4.1)

x	AS	CE in UGNS	AEUGNS
0.1	0.97044553354850817693	0.94176453358424870951	3.0000×10^{-20}
0.2	0.94176453358424870954	0.91393118527122818673	2.0000×10^{-20}
0.3	0.91393118527122818675	0.88692043671715751552	1.0000×10^{-20}
0.4	0.88692043671715751553	0.88692043671715751553	0.0000×10^{00}
0.5	0.86070797642505780723	0.86070797642505780723	0.0000×10^{00}
0.6	0.83527021141127202131	0.83527021141127202131	0.0000×10^{00}
0.7	0.81058424597018709984	0.81058424597018709984	0.0000×10^{00}
0.8	0.78662786106655340922	0.78662786106655340922	0.0000×10^{00}
0.9	0.76337949433685318568	0.76337949433685318568	0.0000×10^{00}
1.0	0.74081822068171786607	0.74081822068171786607	0.0000×10^{00}
1.1	0.71892373343192616956	0.71892373343192616956	0.0000×10^{00}
1.2	0.69767632607103105721	0.69767632607103105721	0.0000×10^{00}
1.3	0.67705687449816469988	0.67705687449816469988	0.0000×10^{00}
1.4	0.65704681981505678160	0.65704681981505678160	0.0000×10^{00}
1.5	0.63762815162177329314	0.63762815162177329314	0.0000×10^{00}

Table 4.2: Numerical Result of Electrical Circuit (Problem 4.2)

x	CC in UGNS
0.1	0.57302939321554260226
0.2	1.49232365250852569160
0.3	2.79931467490199009310
0.4	4.81355101972362053890
0.5	7.75400368779556122280
0.6	12.20507845829009638100
0.7	18.78086687394214730300
0.8	28.65522529689488181600
0.9	43.32160220759823010900
1.0	65.26570036592080659800
1.1	97.93801310795109536900
1.2	146.74381104359110019000
1.3	219.48907093404853959000
1.4	328.07668113298548410000
1.5	490.00592502758306632000

Table 4.3: Numerical Result of Stiff Differential Equation (Problem 4.3) using y_1

x	AS	NS	AEUGNS	AES17	AES18
0.1	0.90483741803595957316	0.90483741803595957090	2.2600×10^{-18}	3.6100×10^{-07}	2.2000×10^{-09}
0.2	0.81873075307798185867	0.81873075307798181707	4.1600×10^{-17}	3.2100×10^{-07}	3.3100×10^{-08}
0.3	0.74081822068171786607	0.74081822068171785725	8.8200×10^{-18}	6.2800×10^{-07}	6.1000×10^{-08}
0.4	0.67032004603563930074	0.67032004603563923033	7.0410×10^{-17}	5.6500×10^{-07}	8.2500×10^{-08}
0.5	0.60653065971263342360	0.60653065971263341966	3.9400×10^{-18}	6.6900×10^{-07}	9.9500×10^{-07}
0.6	0.54881163609402643263	0.54881163609402637807	5.4560×10^{-17}	6.0300×10^{-07}	1.1300×10^{-07}
0.7	0.49658530379140951470	0.49658530379140951520	5.0000×10^{-19}	5.9200×10^{-07}	1.2200×10^{-07}
0.8	0.44932896411722159143	0.44932896411722158452	6.9100×10^{-18}	5.3600×10^{-07}	1.2900×10^{-07}
0.9	0.40656965974059911188	0.40656965974059911970	7.8200×10^{-17}	7.3800×10^{-07}	1.3200×10^{-07}
1.0	0.36787944117144232160	0.36787944117144234344	2.1840×10^{-17}	6.7000×10^{-07}	1.3500×10^{-07}

Table 4.4: Numerical Result of Stiff Differential Equation (Problem 4.3) using y_2

x	AS	NS	AEUGNS	AES17	AES18
0.1	-0.90483741803595957316	-0.90483741803595957266	5.0000×10^{-19}	3.6000×10^{-07}	3.1700×10^{-08}
0.2	-0.81873075307798185867	-0.81873075307798177968	7.8990×10^{-17}	3.3000×10^{-07}	1.0000×10^{-09}
0.3	-0.74081822068171786607	-0.74081822068171785928	6.7900×10^{-18}	6.2700×10^{-07}	3.2700×10^{-08}
0.4	-0.67032004603563930074	-0.67032004603563915972	1.4102×10^{-16}	5.6500×10^{-07}	5.6300×10^{-08}
0.5	-0.60653065971263342360	-0.60653065971263342127	2.3300×10^{-18}	6.6800×10^{-07}	7.5500×10^{-08}
0.6	-0.54881163609402643263	-0.54881163609402632051	1.1212×10^{-16}	6.0200×10^{-07}	9.1200×10^{-08}
0.7	-0.49658530379140951470	-0.49658530379140951582	1.1200×10^{-18}	5.9200×10^{-07}	1.0400×10^{-07}
0.8	-0.44932896411722159143	-0.44932896411722157172	1.9710×10^{-17}	5.3700×10^{-07}	1.1100×10^{-07}
0.9	-0.40656965974059911188	-0.40656965974059911920	7.3200×10^{-18}	7.3800×10^{-07}	1.1700×10^{-07}
1.0	-0.36787944117144232160	-0.36787944117144235862	3.7020×10^{-17}	6.7000×10^{-07}	1.2100×10^{-07}

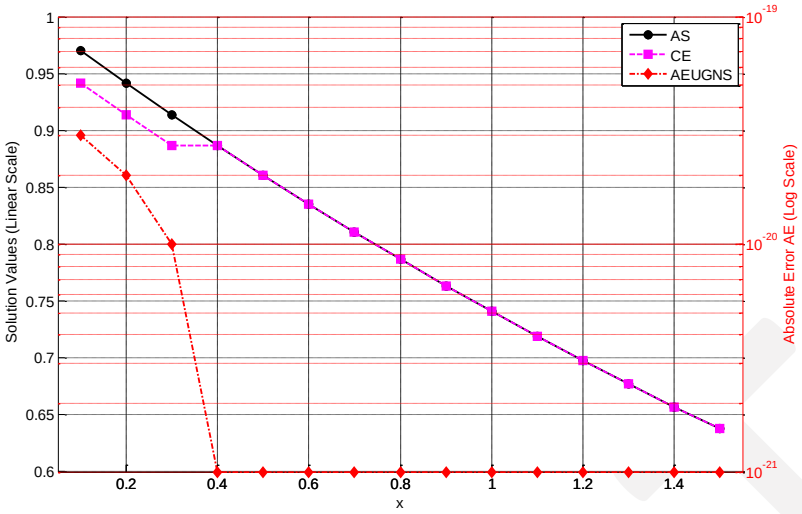

Figure 4.1: Graphical Curve of Table 4.1

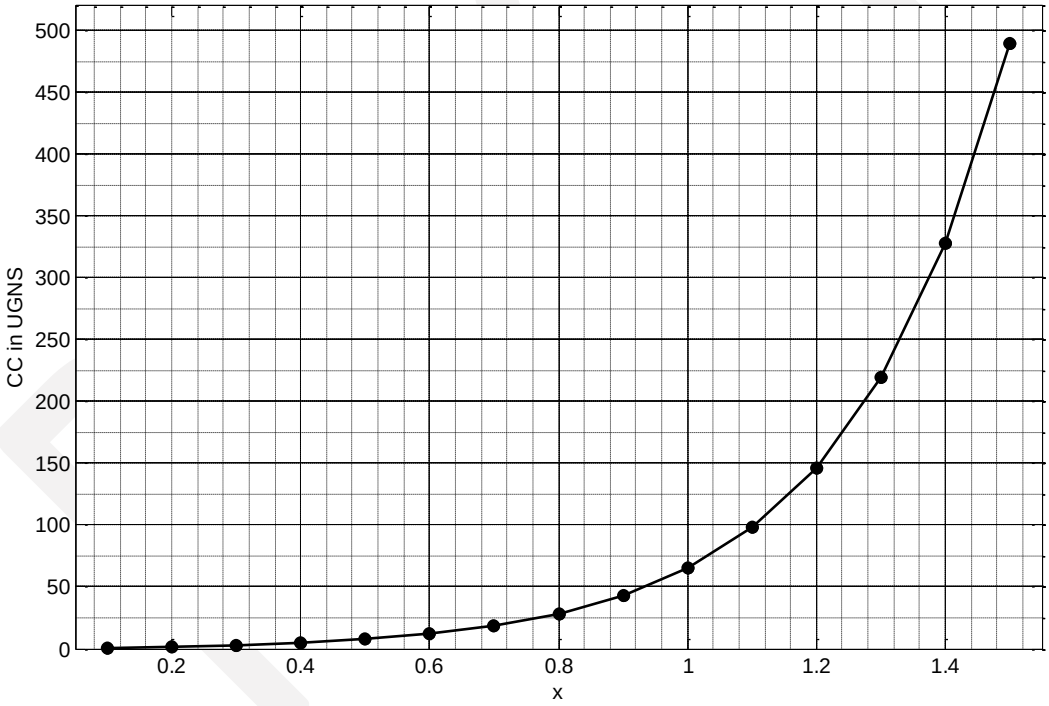

Figure 4.2: Graphical Curve of Table 4.2

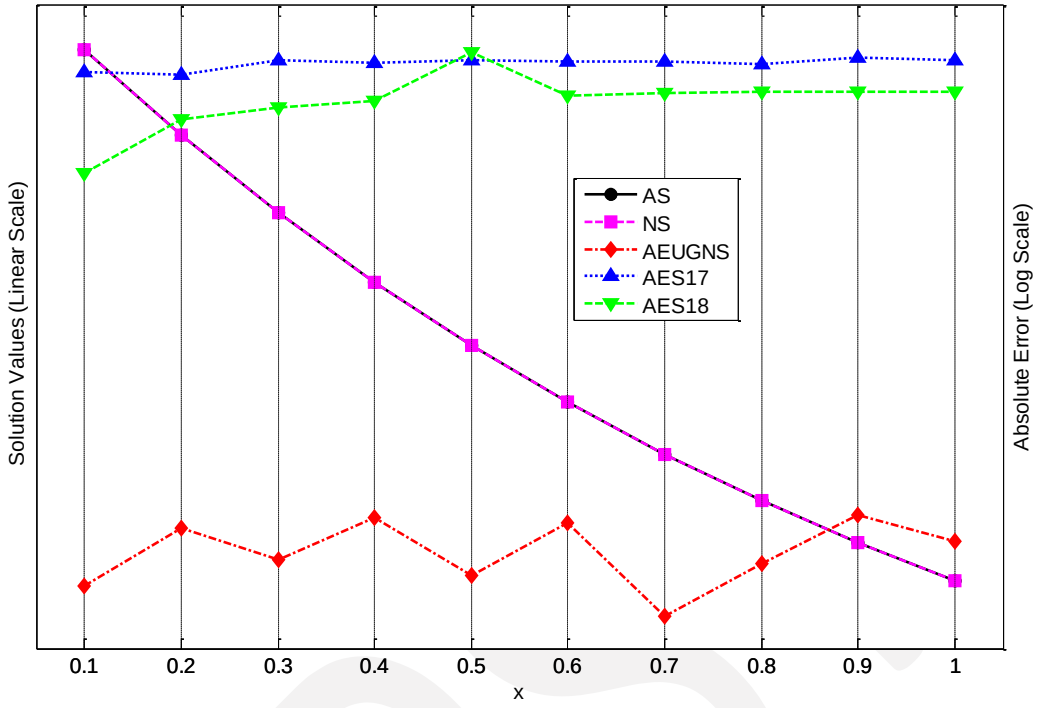

Figure 4.3: Graphical Curve of Table 4.3

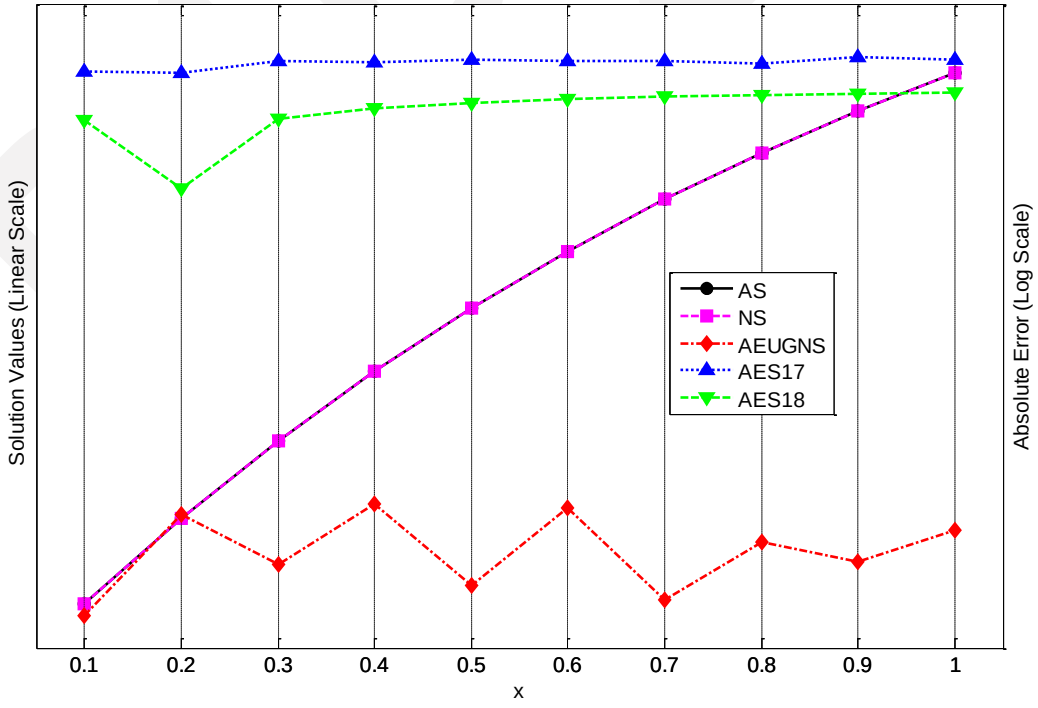

Figure 4.4: Graphical Curve of Table 4.4

5 Discussion of Results

The numerical formulation of the Uniform Grid Numerical Scheme (UGNS) demonstrates a systematic derivation using a power series polynomial as the fundamental function, leading to a continuous linear multistep method. Through interpolation at selected points and collocation of the first and second derivatives, the scheme is expressed in matrix form and solved via Gaussian elimination, ultimately yielding a discrete UGNS presented in equation (2.6). The analysis of the scheme's properties confirms its robustness: it is of uniform order six with an error constant, consistent according to the conditions outlined in Definition 3.2, and zero-stable as verified by the root condition of the first characteristic polynomial. These properties collectively satisfy Dahlquist's theorem, ensuring the convergence of the UGNS.

Furthermore, the absolute stability region of the UGNS is determined using the Boundary Locus Method, and the resulting stability polynomial is plotted in Figure 3.1. The region of absolute stability, defined in the complex plane where the numerical solution remains bounded, confirms the scheme's suitability for stable integration. The UGNS exhibits desirable numerical characteristics high order, consistency, zero-stability, and a well-defined stability region making it a reliable and convergent method for solving differential equations.

The Uniform Grid Numerical Scheme (UGNS) was tested across three distinct problems drawn from industrial chemistry, applied physics, and applied mathematics, and the results demonstrate consistently strong performance. In the ester hydrolysis problem (Table 4.1, Figure 4.1), the scheme reproduced the exponential decay of ester concentration with essentially perfect agreement with the analytical solution. From the early time points onward, the numerical values matched the exact solution to the full precision displayed, and from $x = 0.4$ to $x = 1.5$ the agreement was complete. Figure 4.1 would show the numerical and analytical curves as visually indistinguishable, confirming that UGNS captures the smooth exponential behavior of first-order chemical kinetics without any observable deviation.

For the RC electrical circuit problem (Table 4.2, Figure 4.2), the UGNS results show the capacitor charge increasing monotonically from 0.573 at t $x = 0.1$ to 490.006 at $x = 1.5$, with accelerating increments characteristic of the early charging phase of an RC circuit. This behavior is physically consistent with the transient response of a series RC circuit, where the charge follows an exponential approach to its steady-state value. Figure 4.2 would display this charging curve as a smooth, concave-up profile. However, because no analytical solution or error column was provided for this problem, the assessment is limited to qualitative physical reasonableness rather than quantitative accuracy, which represents a gap in the validation compared with the other two problems. The stiff differential equation problem (Tables 4.3 and 4.4, Figures 4.3 and 4.4) provides the most compelling evidence of the scheme's capabilities, particularly through direct comparison with existing methods. For both the positive and negative exponential solution branches, UGNS maintained accuracy at the level of machine precision across all time points, while the competing methods of Skwame et al. (2017) and Skwame (2018) produced errors many orders of magnitude larger. This superiority held consistently for both branches and for the different step sizes tested, indicating that the scheme handles stiffness robustly without the stability difficulties that often plague explicit methods. Figures 4.3 and 4.4 would show the numerical and analytical curves coinciding completely, visually reinforcing the tabulated agreement.

Taken together, the results establish UGNS as a highly accurate, stable, and versatile scheme. It achieves near-exact agreement with analytical solutions for linear non-stiff problems, handles both exponential growth and decay in stiff systems with equal facility, and substantially outperforms established competing methods on the stiff benchmark. The scheme's ability to maintain machine-

precision accuracy across different problem types, solution behaviors, and step sizes is a significant strength. The principal limitation is the absence of quantitative error analysis for the RC circuit problem, and the validation would be further strengthened by a systematic convergence study and a discussion of computational cost; nevertheless, the evidence presented strongly supports the effectiveness and superiority of the Uniform Grid Numerical Scheme.

6 Summary and Conclusion

This study successfully developed and analyzed a Uniform Grid Numerical Scheme (UGNS) for the numerical solution of first-order ordinary differential equations, particularly stiff initial value problems. The scheme was derived using a power series polynomial as the basis function, with interpolation and collocation procedures incorporating second derivative information to yield a continuous linear multistep method. The theoretical analysis confirmed that the UGNS possesses desirable numerical properties: it is of uniform order six, consistent, zero-stable, and convergent according to Dahlquist's theorem. The region of absolute stability, determined via the Boundary Locus Method, further established the scheme's suitability for stable integration. The validation of the scheme on three benchmark problems ester hydrolysis, an RC electrical circuit, and a stiff system of differential equations demonstrated its exceptional performance. The UGNS achieved near-exact agreement with analytical solutions for the non-stiff chemical kinetics problem, maintained machine-precision accuracy for the stiff and substantially outperformed existing methods from the literature. The graphical comparisons visually reinforced these findings, with numerical and analytical curves appearing indistinguishable.

In conclusion, the Uniform Grid Numerical Scheme represents a significant advancement in the numerical treatment of ordinary differential equations, particularly for stiff systems where conventional explicit methods often suffer from stability constraints. The scheme's ability to maintain high accuracy across diverse problem types ranging from exponential decay in chemical kinetics to the transient response of electrical circuits and the challenging behavior of stiff systems demonstrates its versatility and robustness. The combination of high order, consistency, zero-stability and a well-defined absolute stability region makes the UGNS a reliable and efficient computational tool for real-world applications in applied science, engineering, and technology. While the validation would be further strengthened by a systematic convergence study, computational cost analysis, and quantitative error assessment for the RC circuit problem, the evidence presented strongly supports the effectiveness and superiority of the UGNS. Future research could explore the extension of the scheme to higher-dimensional systems, boundary value problems, and partial differential equations, as well as its application to more complex real-world models in fluid dynamics, population dynamics, and control systems.

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